On soliton and other travelling-wave solutions of a perturbed sine-Gordon equation

Gaetano Fiore[1,2]

[1]Dip. di Matematica e Applicazioni, Università "Federico II",
V. Claudio 21, 80125 Napoli
[2]I.N.F.N., Sezione di Napoli,
Complesso MSA, V. Cintia, 80126 Napoli

Abstract

We give an exhaustive, non-perturbative classification of exact travelling-wave solutions of a perturbed sine-Gordon equation (on the real line or on the circle) which is used to describe the Josephson effect in the theory of superconductors and other remarkable physical phenomena. The perturbation of the equation consists of a constant forcing term and a linear dissipative term. On the real line stable solutions with bounded energy density are either the constant one, or of solitonic (or kink) type, or of array-of-solitons type, or of "half-array-of-solitons" type. While the first three have unperturbed analogs, the last type is essentially new. We also propose a convergent method of successive approximations of the (anti)soliton solution.

1 Introduction

The purpose of this work is an exhaustive, non-perturbative analysis of travelling-wave solutions of the "perturbed" sine-Gordon equation

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi + \alpha\varphi_t + \gamma = 0, \qquad x \in \mathbb{R}, \tag{1}$$

for *all* constant $\alpha \geq 0, \gamma \in \mathbb{R}$. This equation has been used to describe with a good approximation a number of interesting physical phenomena, notably Josephson effect in the theory of superconductors [11], which is at the base [4] of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3-6 in [5]), or more recently also the propagation of localized magnetohydrodynamic modes in plasma physics [22]. The last two terms are respectively a dissipative and a forcing one; the sine-Gordon equation (sGe) is obtained by setting them equal to zero.

In the Josephson effect (for an introduction see e.g. Chp 1 in [4]) $\varphi(x,t)$ is the phase difference of the macroscopic quantum wavefunctions describing the Bose-Einstein condensates of Cooper pairs in two superconductors separated by a very thin, narrow and long dielectric (a socalled "Josephson junction"). The γ term is the (external) "bias current", providing energy to the system, whereas the dissipative term $\alpha\varphi_t$ is due to Joule effect of the residual current across the junction due to single electrons.

It important to clarify: a) which solutions of the sGe are deformed into solutions of (1) with the same qualitative features; b) whether (1) admits also new kinds of solutions. Candidate approximations to the former can be obtained within the standard perturbative method [12, 18, 13, 14] based on modulations of the unperturbed solutions with slowly varying parameters (typically velocity, space/time phases, etc.) and small radiation components. In particular, the Ansatz for a deformation of a travelling-wave solution $\varphi_{(0)}(x,t) = g_{(0)}(x - vt)$ of the sGe reads

$$\varphi(x,t) = g_{(0)}\Big(x - x_0(t) - \tilde{v}(t)t\Big) + \gamma\varphi_{(1)}(x,t) + \ldots; \tag{2}$$

γ plays the role of perturbation parameter, whereas the slowly varying $x_0(t), \tilde{v}(t)$ and the perturbative "radiative" corrections $\gamma\varphi_{(1)}(x,t) + \ldots$ have to be computed perturbatively in terms of $\alpha\varphi_t + \gamma$. If in particular $\varphi_{(0)}(x,t)$ is a (anti)solitonic [or (anti)kink] solution, one finds [8, 15] also candidate approximate solutions with *constant* velocity

$$\tilde{v}(t) \equiv v_\infty := \pm[1 + (4\alpha/\pi\gamma)^2]^{-\frac{1}{2}} \tag{3}$$

which are characterized by a power balance between the dissipative term $\alpha\varphi_t$ and the external force term γ. Clearly the perturbative series (2) will converge to an exact solution *only within its ray of convergence*, admitted that the latter is nonzero. Even if this were the case and one were able to control the convergence, this perturbative approach would certainly fail for large γ. Numerical resolutions [10, 16] of (1) are surely a useful alternative, but cannot provide solid, exhaustive answers to the two questions above.

The purpose of this work is to answer questions a), b) by providing (section 3) a non-perturbative *classification of exact travelling-wave* solutions of (1) on the real line or on the circle *for all* $\alpha \geq 0, \gamma \in \mathbb{R}$, and to propose (section 4) an improved method of

successive approximations converging to the (anti)soliton solutions, at least for sufficiently small γ. We stick to solutions of physical interest, namely solutions that are stable and have bounded energy density h, and therefore also bounded derivatives; in the sequel we shall denote them as the *relevant solutions*. We have begun this job in Ref. [7]. The classification is based on a detailed phase space analysis (initiated in [23]) of the solutions of the o.d.e. (ordinary differential equation) with varying parameters derived (section 2) by replacing in the p.d.e. (partial differential equation) (1) a function of a suitable combination of x, t only (travelling-wave Ansatz), and on the comparisons of the solutions for different values of the parameters; although this seems a rather natural thing to do, we have not found in the literature such a classification. If the velocity is ± 1 the o.d.e. is of first order and can be solved by quadrature, otherwise it is the second order one describing the motion along a line of a particle subject to a "washboard" potential and immersed in a linearly viscous fluid, and therefore the problem is essentially reduced to studying this simpler mechanical analog. A number of useful monotonicity properties (section 2.1) allow us in particular to identify (Theorem 1 in section 3) four families of relevant solutions: three of them (the arrays of solitons for all values of γ, the solitons and the constants only for $\gamma < 1$) are deformations of analogous families of solutions of the sGe, whereas the fourth family is without unperturbed analog: as each of its elements interpolates between a soliton and an array of solitons (see Fig. 3), we have called it a "half-array of solitons". The stability of these solutions has been tested numerically. No other relevant solutions exist [7]. The families of perturbed solitons and arrays of solitons depend on one free parameter less than the unperturbed ones, as the propagation velocity v turns out to be a function of α, γ, which for the soliton coincides, at lowest order in γ, with (3).

1.1 Preliminary considerations

Space or time translations transform any solution into a two-parameter family of solutions; one can choose any of them as the family representative element.

The sGe describes also the dynamics of the continuum limit of a sequence of neighbouring heavy pendula constrained to rotate around the same horizontal x-axis and coupled to each other through a torque spring [21]; $\varphi(x,t)$ is the deviation angle from the lower vertical position at time t of the pendulum having position x. One can model also the dissipative term $-\alpha\varphi_t$ of (1) by immersing the pendula in a linearly viscous fluid, and the forcing term γ by assuming that a uniform, constant torque distribution is applied to the pendula. This mechanical analog allows a qualitative comprehension of the main features of the solutions, e.g. of their instabilities. The constant solutions of (1) are $\varphi^s(x,t) \equiv -\sin^{-1}\gamma + 2\pi k$ and $\varphi^u(x,t) \equiv \sin^{-1}\gamma + (2k+1)\pi$. The former are stable, the latter unstable, as they yield respectively local minima and maxima of the energy density

$$h := \frac{\varphi_t^2}{2} + \frac{\varphi_x^2}{2} + \gamma\varphi - \cos\varphi. \tag{4}$$

In the mechanical analog they respectively correspond to configurations with all pendula hanging down or standing up.

Our *definition of a soliton solution* φ is: φ is a non-constant *stable travelling-wave solution with* φ_x, φ_t *rapidly going to zero outside some localized region.* Then mod. 2π it

must be

$$\lim_{x\to-\infty}\varphi(x,t)=-\sin^{-1}\gamma,\qquad \lim_{x\to+\infty}\varphi(x,t)=-\sin^{-1}\gamma+2n\pi \tag{5}$$

with $n\in\mathbb{Z}$. As we shall recall below, only $n=1$ (soliton or kink) and $n=-1$ (antisoliton or antikink) are possible [whereas $n=0$ corresponds to the constant φ_s]. In the mentioned mechanical model the (anti)solitonic solution describes a localized twisting of the pendula chain by 2π (anti)clockwise around the axis, moving with constant velocity. The above condition yields an energy density h (rapidly) going to two local minima as $x\to\pm\infty$. Although this makes the total Hamiltonian $H:=\int_{-\infty}^{+\infty}h(x,t)dx$ divergent, the time-derivative is finite and non-positive:

$$\dot{H}=-\int_{-\infty}^{\infty}\alpha\varphi_t^2dx\le 0.$$

[The negative sign at the rhs shows the dissipative character of the time derivative term in (1)]. The effect of $\gamma\neq 0$ is to make the values of the energy potential density at any two minima different; this leaves room for an indefinite compensation of the dissipative power loss by a falling down in the total potential energy from one minimum to the lower next, and so may account for solutions not being damped to constants as $t\to\infty$.

Without loss of generality we can assume $\gamma\ge 0$. If originally this is not the case, one just needs to replace $\varphi\to-\varphi$. If $\gamma>1$ no solutions φ having finite limits and vanishing derivatives for $x\to\pm\infty$ can exist, in particular no static solutions. If $\gamma=1$ the only static solution φ having for $x\to\pm\infty$ finite limits and vanishing derivatives is $\varphi\equiv-\pi/2$ (mod 2π), which however is manifestly unstable.

2 Preliminary stages of the analysis

We specify our travelling wave Ansatz as follows:

$$\begin{array}{lll}
\xi:=\pm x-t & \varphi(x,t)=g(\xi)-\pi & \text{if } v=\pm1,\\
\xi:=-\mathrm{sign}(v)\frac{x-vt}{\sqrt{v^2-1}} & \varphi(x,t)=-g(\xi) & \text{if } v^2>1,\\
\xi:=\mathrm{sign}(v)\frac{x-vt}{\sqrt{1-v^2}} & \varphi(x,t)=g(\xi)-\pi & \text{if } 0<v^2<1,\\
\xi:=x & \varphi(x,t)=g(\xi)-\pi & \text{if } v=0.
\end{array} \tag{6}$$

If $v=\pm1$, replacing the Ansatz in (1) one obtains the **first order o.d.e.**,

$$\alpha g'=\gamma-\sin g. \tag{7}$$

We have already argued in [7] that if $\gamma<1$ all its solutions yield [7] unstable solutions of (1), except the static constant one $\varphi^s(x,t)\equiv-\sin^{-1}\gamma$. The same argument holds also if $\gamma=1$. If $\gamma>1$, by integrating one finds

$$\xi-\xi_0=\int_{\xi_0}^{\xi}d\xi'=\alpha\int_{g_0}^{g}\frac{ds}{\gamma-\sin s};$$

the denominator is positive for all $s \in \mathbb{R}$, so that the solution g is strictly monotonic and linear-periodic, i.e. the sum of a linear and a periodic function, so that

$$g(\xi + \Xi) = g(\xi) + 2\pi, \qquad\qquad \Xi := \int_0^{2\pi} \frac{ds}{u(s)} \tag{8}$$

with $u(g) := g'(g) = (\gamma - \sin g)/\alpha$; this will yield (Theorem 1) a stable solution $\check{\varphi}$ of (1), representing an 'array of (anti)solitons' travelling with velocity ± 1 (such velocities are not possible in the sine-Gordon case).

In the rest of the section we assume that $v \neq \pm 1$. Replacing in (1) we find in all three remaining cases the **second order** o.d.e.

$$g'' + \mu g' + U_g(g) = 0, \qquad \xi \in \mathbb{R}, \tag{9}$$

which can be regarded as the 1-dimensional equation of motion w.r.t. the 'time' ξ of a particle with unit mass, position g, subject to a 'washboard' potential energy' $U(g)$ and a viscous force with viscosity coefficient given by

$$U(g) := -(\cos g + \gamma g), \qquad\qquad \mu := \frac{\alpha}{\sqrt{|v^{-2} - 1|}}. \tag{10}$$

Note that in equation (9) α, v appear only through their combination $(10)_2$, and that in the range $|v| \in [0, 1[$ (resp. $|v| \in]1, \infty[$) $\mu(|v|)$ is strictly increasing (resp. decreasing), and therefore invertible. In Fig. 1 $U(g)$ is plotted for four different values of γ; it admits local minima (resp. maxima) only if $0 \leq \gamma < 1$, in the points

$$g_k^m := \sin^{-1}\gamma + 2k\pi, \qquad\qquad (\text{resp. } g_k^M := -\sin^{-1}\gamma + (2k+1)\pi).$$

As $\gamma \to 1$ the points g_k^m, g_k^M approach each other, and for $\gamma = 1$ $g_k^m = g_k^M = (2k + 1/2)\pi$ are inflections points. For $\gamma > 1$ no minima, maxima or inflections exist, and $U_g < 0$ everywhere. The "total energy of the particle" $\mathrm{e} := g'^2/2 + U(g)$ is a non-increasing function of ξ, as $\mathrm{e}' = -\mu g'^2$.

An exhaustive classification of the solutions of equation (9) for all values of μ, γ has been performed long ago in several works, starting from [23, 1] (see [17] or [2] for comprehensive presentations). The equation is equivalent to the autonomous first order system

$$\begin{aligned} u' &= -\mu u - \sin g + \gamma, \\ g' &= u. \end{aligned} \tag{11}$$

Since the rhs's are functions of g, u with bounded continuous derivatives, by the Peano-Picard theorem on the extension of the integrals all solutions are defined (global existence) on all $-\infty < \xi < \infty$, and the paths [i.e. the trajectories in the phase space (g, u)] do not intersect (uniqueness). Each is uniquely identified by any of its points (g_0, u_0). As known, the paths may have finite endpoints (limits as $\xi \to \pm\infty$) only at singular points, i.e. points where the rhs's(11) vanish. These exist only for $\gamma \leq 1$ and are

$$\begin{array}{lll} \text{saddles } A_k = (g_k^M, 0), & \text{nodes, foci or centers } B_k = (g_k^m, 0), & \gamma < 1 \\ \text{saddle-nodes } C_k = ((2k + 1/2)\pi, 0) & & \gamma = 1 \end{array} \tag{12}$$

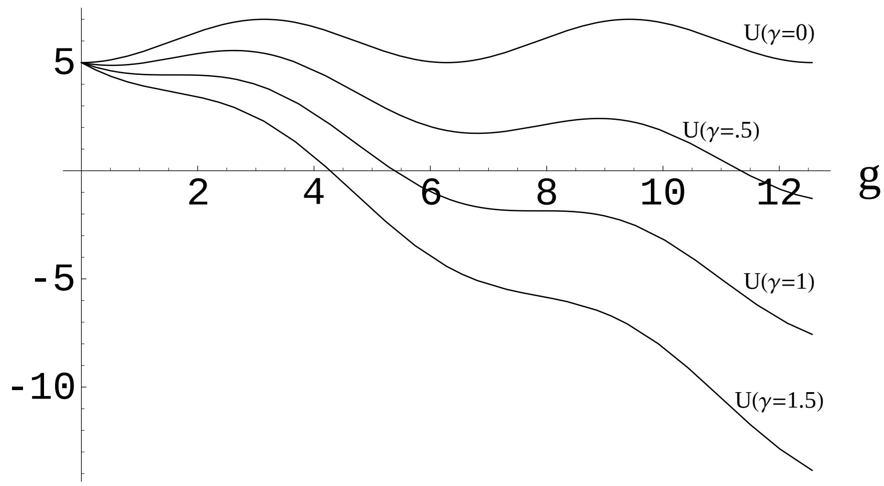

Figure 1: The potential energy $U = 6-(\cos g+\gamma g)$ for $\gamma = 0$, $\gamma = .5$, $\gamma = 1$, $\gamma = 1.5$.

Their classification is recalled in Appendix 4.1. Finally, the solutions are continuous functions of the parameters μ, γ and of (g_0, u_0) (away from singular points), uniformly in every compact subset.

We are going to see that the latter dependences are also monotonic. To analyze them and the asymptotic behaviour of the paths near the endpoints it is useful to eliminate the 'time' ξ and adopt as an independent variable the 'position' g, as in the unperturbed case. The path of any solution $g(\xi)$ of (9) is cut into pieces by the axis $u = 0$. Let $X \equiv]\xi_-, \xi_+[\subseteq \mathbb{R}$ be the 'time' interval corresponding to a piece,

$$\epsilon := \text{sign}(u(\xi)), \qquad\qquad \xi \in X$$

be its sign and let $G \equiv]g_-, g_+[:= g(X)$. In X the function $g(\xi)$ can be inverted to give a function $\xi : g \in G \to \xi(g) \in X$. So one can express the 'velocity' u and the 'kinetic energy' $z := u^2/2$ of the 'particle' as functions of its 'position' g. By derivation we find that $g''(\xi) = u_g(g(\xi))g'(\xi)$ and the second order problem (9) with initial condition $(g(\xi_0), u(\xi_0)) = (g_0, u_0)$ in X is equivalent to two **first order problems**: the first is

$$z_g(g) + \epsilon\mu\sqrt{z(g)} + \sin g - \gamma = uu_g(g) + \mu u(g) + \sin g - \gamma = 0, \qquad u(g_0) = u_0 \tag{13}$$

(note that this is invariant under the replacement $g \to g + 2\pi$), which has to be solved first, and yields a solution $u = u(g; g_0, u_0; \mu, \gamma)$ continuous in all arguments (away from singular points); the second is

$$g'(\xi) = u(g(\xi)), \qquad\qquad g(\xi_0) = g_0, \tag{14}$$

is integrated out by quadrature

$$\xi - \xi_0 = \int_{\xi_0}^{\xi} d\xi' = \int_{g_0}^{g} \frac{ds}{u(s)} = \epsilon \int_{g_0}^{g} \frac{ds}{\sqrt{2z(s)}} \tag{15}$$

and implicitly yields a solution $g = g(\xi; g_0, u_0; \mu, \gamma)$ in X. If X is not the whole $\mathbb{R}$, the final step is the patching of solutions in adjacent intervals X.

Choosing in (15) g as $g_\pm$ one obtains $\xi_\pm$. If $z(g)$ vanishes as $\eta^a := |g_\pm - g|^a$ with $a \geq 2$ as $g \uparrow g_+$ or $g \downarrow g_-$, then $\xi_+ = \infty$ or $\xi_- = -\infty$. The behaviour of $u(g), z(g)$ near $g_\pm$ can be determined immediately solving (13) at leading order in a left (resp. right) neighbourhood of g_+ (resp. g_-). In particular, if $\gamma < 1$ and $g_\pm = g_k^M$ (a maximum point of U) then the equation obtained by replacing the power law Ansatz $u(g) = \eta^{a/2} u_\pm + o(\eta^{a/2}$ in (13) is solved by

$$
\begin{aligned}
u(g) &\approx (g_+ - g) u_{+\epsilon} \qquad \text{as } g \uparrow g_+, \\
u(g) &\approx (g - g_-) u_{-\epsilon} \qquad \text{as } g \downarrow g_-,
\end{aligned}
\tag{16}
$$

where for $\epsilon, \epsilon' \in \{+, -\}$ $u_{\epsilon'\epsilon}$ is defined by

$$
u_{\epsilon'\epsilon} := \frac{1}{2}\left(\epsilon'\mu + \epsilon\sqrt{\mu^2 + 4\sqrt{1-\gamma^2}}\right).
$$

Formula (16) gives the leading behaviour of the four separatrices having an end on A_k.

Problem (13) is also equivalent to the **Volterra-type integral equation**

$$
z(g) = z_0 + U(g_0) - U(g) - \epsilon \int_{g_0}^{g} ds\, \mu \sqrt{2z(s)} \tag{17}
$$

where $z_0 := u_0^2/2$. When $\mu = 0$ (no dissipation) this gives the solutions explicitly and amounts to the conservation of the total energy of the 'particle' $\mathrm{e}(g) = z(g) + U(g)$.

2.1 Monotonicity properties

In agreement with the physical intuition, **the solutions of (13) and the extremes of G depend on the parameters μ, z_0, γ monotonically**. In the Appendix we prove the following propositions, which include and extend the results of [23, 1].

Proposition 1 *As functions of z_0: $z = u^2/2$ is strictly increasing; g_+ is increasing and g_- decreasing (strictly as long as they have not reached the values $\pm\infty$).*

Proposition 2 *As a function of both $\mu, -\epsilon\gamma$ the solution $u(g; g_0, u_0; \mu, \gamma)$ is strictly decreasing (resp. strictly increasing) for $g \in]g_0, g_+[$ (resp. $g \in]g_-, g_0[$). Correspondingly, the solution $g(\xi; g_0, u_0; \mu, \gamma)$ is strictly decreasing as a function of both $\epsilon\mu, -\gamma$, and so is either extreme $g_\pm$ (strictly as long as it has not reached values $\pm\infty$).*

Remark. In general $g_\pm$ will be discontinuous functions of μ, z_0, γ at $g_\pm = g_k^M$.

Whenever the domain G of the solution $z(g)$ contains a whole interval $]g, g+2\pi[$ we define

$$
I(z, g) := \int_{g}^{g+2\pi} ds \sqrt{2z(s)} \tag{18}
$$

Given any $g_0 \in \bar{G}$, let $g_k := g_0 + 2\pi k$, $K := \{k \in \mathbb{Z} \mid g_k \in \bar{G}\}$ and $I_k := I(z, g_k)$ if $k, k+1 \in K$.

Proposition 3 *If $\epsilon = -$ the sequences $\{z(g_k)\}, \{I_k\}$ are strictly increasing and diverging as $k \to \infty$, with K bounded from below. If $\epsilon = +$ the sequences $\{z(g_k)\}, \{I_k\}$ are: either constant, with $K = \mathbb{Z}$; or strictly increasing and converging as $k \to \infty$, with K bounded from below; or strictly decreasing, diverging as $k \to -\infty$, and either converging as $k \to \infty$, or with K upper bounded. Moreover,*

$$z(g_{k+1}) - z(g_k) = 2\pi\gamma - \epsilon\mu I_k. \tag{19}$$

3 The relevant solutions

If $\gamma = \alpha = \mu = 0$ (sGe) the total energy e of the 'particle' is conserved and its value (together with the free parameter v) parametrizes different kinds of solutions of (9) [3, 6]. Plotting $U(g)$ (Fig. 2 left) we get an immediate qualitative understanding of them. They all have bounded z and therefore g'. This implies that also the corresponding φ_x, φ_t, h are bounded functions of x, t.

If $\gamma, \alpha, \mu \neq 0$ (perturbed sine-Gordon) there are [7] solutions $g(\xi)$ with g' (and therefore also φ_x, φ_t, h) diverging at space and time infinity[1]; as said, they are to be discarded. In Ref. [7] we have analyzed all the possibilities for $\gamma < 1$ and shown (Prop. 1) that *relevant* (in the sense of the introduction) solutions φ, *if they exist, can be* only of four types, all with $v^2 \leq 1$ and $\epsilon := \mathrm{sign}(g') \geq 0$; out of them three are deformations of travelling-wave solutions of the sGe. Actually a simple inspection shows that the arguments and the conclusions given there continue to hold for $\gamma \geq 1$; the analysis could be simplified with the help of the monotonicity properties of section 2.1. Step by step, we are now going to see (Theorem 1) that all four types *actually exist*.

We have already given in the introduction the **constant solutions** φ^s, φ^u. They correspond to $g(\xi) \equiv g_k^M, g_k^m$ respectively, implying that e takes the value of a maximum or minimum of $U(g)$ ($\mathrm{e} = \pm 1$ if $\gamma = 0$), respectively.

If $\gamma = \alpha = \mu = 0$ solutions corresponding to $\mathrm{e} = \mathrm{e}_1 \in]-1, 1[$ are unstable [19, 3, 6]; their deformations for $\gamma, \alpha \neq 0$ are not relevant as well [7].

The (anti)soliton solutions are obtained from non-constant solutions $\hat{g}(\xi)$ having finite limits at both $\xi \to \pm\infty$. The corresponding paths are heteroclinic orbits ending at two neighbouring saddle points, e.g. A_0, A_1: the 'particle', confined in the interval $g_0^M < g < g_1^M$, starts at 'time' $\xi = -\infty$ from one extreme and reaches the other one at $\xi = \infty$ (the extremes being maximum points of $U(g)$). The corresponding 'kinetic energy' $\hat{z}(g)$ will be defined in the same interval and fulfill the boundary conditions

$$\lim_{g \downarrow g_0^M} \hat{z}(g) = 0, \qquad\qquad \lim_{g \uparrow g_1^M} \hat{z}(g) = 0. \tag{20}$$

If $\gamma = \alpha = \mu = 0$ these are characterized by $\mathrm{e} = \hat{\mathrm{e}} \equiv 1$, so that $\hat{g}(\xi) \to \pm\pi$ as $\xi \to \pm\infty$. Explicitly, these two solutions can be obtained from (17), (15) inverting $\xi(g)$. Replacing

[1] For instance, by Prop. 3 if $\epsilon = -$ and $z(g)$ is defined at least in an interval of length 2π then $g_+ = \infty$, $z(g)$ diverges as $g \to \infty$, $g'(\xi), \varphi_x, \varphi_t$ diverge as $\xi \to -\infty$.

the result in (6), mod. 2π they translate into unstable solutions of the sGe if $v^2>1$ (all pendula stand upwards outside a small region), and the celebrated families

$$\hat{\varphi}^{\pm}_{(0)}(x,t;v)=4\tan^{-1}\left\{\exp\left[\pm\frac{x-vt}{\sqrt{1-v^2}}\right]\right\} \tag{21}$$

of stable [19, 3, 6] solutions if $v^2<1$: all pendula of the chain hang downwards outside a small region travelling with velocity v; within that region they twist $n=\pm1$ times, i.e. once clockwise or anti-clockwise, around the x-axis. $\hat{\varphi}^{+}_{(0)}(x,t;v)$ is the one-parameter family of soliton solutions, $\hat{\varphi}^{-}_{(0)}(x,t;v)$ the one-parameter family of antisoliton solutions, the parameter being their velocity v, which can take any value in $]-1,1[$.

If $\gamma>0$ an heteroclinic orbit exists only if $\gamma<1$, $\epsilon=+$ and μ is adjusted to a special value $\hat{\mu}(\gamma)>0$. In fact, consider equation (13) with varying $\mu\geq0$ and impose $(20)_1$. For $\mu=0$ the total energy e is conserved: the corresponding solution $z(g)$ is therefore defined in $g_1:=g_0+2\pi\equiv g_0^M$ and $z(g_1)=2\pi\gamma>0$, whereas for sufficiently large μ [23], $\mu\geq\tilde{\mu}$, the corresponding $z(g)$ is not defined in g_1 (as $g_-<g_1$). By continuity in μ and Prop. 2, there exists a unique $\hat{\mu}(\gamma)>0$ such that the corresponding solution $\hat{z}(g)$) is defined in $G=]g_0^M,g_1^M[$, i.e. fulfills also $(20)_2$ [see Fig. 2 right, where also $\hat{\mathrm{e}}(g)$ is plotted]. By (19) relation (20) implies $\epsilon=+$ and

$$\hat{\mu}\,I(\hat{z},g_0^M)=2\pi\gamma, \tag{22}$$

which is the **energy balance condition** between the energy dissipated by the viscous force and the potential energy gap after a 2π displacement of the 'particle'. Replacing $\hat{z}(g)$ in (15) one finds the mentioned $\hat{g}(\xi)$ (this will have $\hat{g}'>0$ everywhere); replacing the latter in (6) one finds **deformed (anti)soliton solutions**, as described in Theorem 1. By inversion of $(10)_2$ the velocity v will no more be a free parameter, but a function (in the range $]-1,1[$) of γ,α and of the helicity $\pm$ of the (anti)soliton solution $\hat{\varphi}^{\pm}$.

If $\gamma=1$ also a path connecting C_k,C_{k+1} can be found [17], but this will be of no use because it yields a manifestly unstable φ.

The "array of (anti)soliton" solutions are obtained from linear-periodic solutions $\check{g}$, in the sense (8): the 'particle' travels towards the right from $g_-=-\infty$ to $g_+=\infty$ (or towards the left from $g_+=\infty$ to $g_-=-\infty$) and its 'kinetic energy' $\check{z}(g)$ is 2π-periodic, in particular takes the same value z_M at all points g_k^M,

$$\check{z}(g_k^M)=z_M \qquad\qquad \forall k\in\mathbb{Z}. \tag{23}$$

If $\gamma=\alpha=\mu=0$ the corresponding solutions $\check{g}_{(0)}(\pm\xi;\mathrm{e})$ can be equivalently characterized by the constant $\mathrm{e}\equiv\check{\mathrm{e}}>1$, or by $z_M=\check{\mathrm{e}}-U(g_k^M)$, or by the period $\Xi_{(0)}$ [because of $(8)_2$]. Again, the corresponding solutions of the sGe are [19, 3, 6] unstable if $v^2>1$ and stable if $v^2<1$ ('most' pendula down in the pendula chain model). The stable solutions $\check{\varphi}^{\pm}_{(0)}(x,t)=\check{g}_{(0)}(\pm\xi)$ respectively describe two-parameter families of evenly spaced "arrays of solitons and antisolitons", the two parameters being their velocity v, which can take any value in $]-1,1[$, and one of the variables $\check{\mathrm{e}},z_M,\Xi_{(0)}$.

If $\gamma>0$, solutions fulfilling (23) exist only if $\epsilon=+$ and μ is adjusted to a special value $\check{\mu}(\gamma,z_M)>0$. In fact, consider problem (13) with varying μ, $z_0\equiv z_M>0$ and set

$$g_0=\begin{cases} g_k^M & \text{if } \gamma<1,\\ 2k\pi-\frac{1}{2}\pi & \text{if } \gamma\geq1,\end{cases}$$

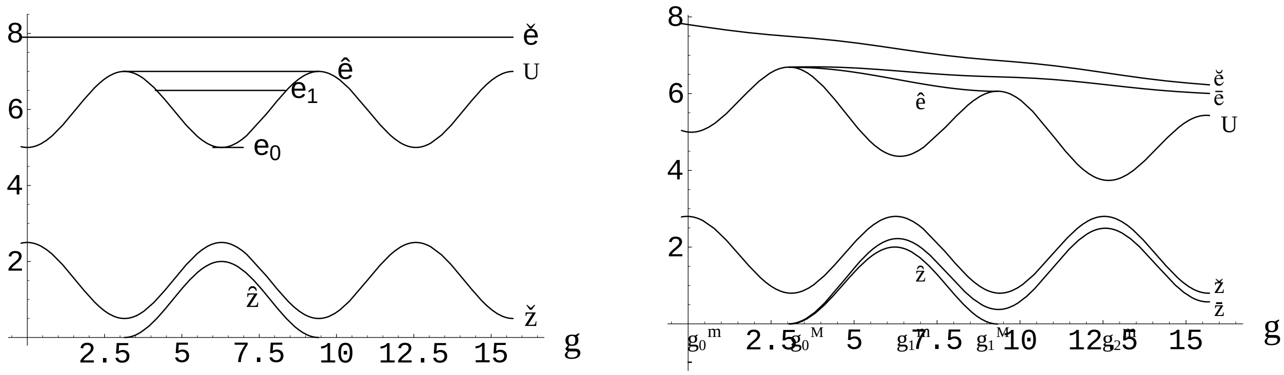

Figure 2: The potential energy $U(g) = 6 - (\cos g + \gamma g)$ for $\gamma = 0$ (left) and $\gamma = .1$ (right). Correspondingly, the 'kinetic energies' and the 'total energies': 1) $\hat{z}, \hat{\mathrm{e}}$ associated to the soliton, $\mu = \hat{\mu}(\gamma)$; 2) $\check{z}, \check{\mathrm{e}}$ associated to an array of solitons, $\mu < \hat{\mu}(\gamma)$; 3) $\bar{z}, \bar{\mathrm{e}}$ associated to a half-array of solitons solutions, $\mu < \hat{\mu}(\gamma)$.

Because of (19) condition (23) implies again $\epsilon = +$. For $\mu = 0$ the total energy e is conserved: the solution $z(g)$ is therefore defined in $g_1 := g_0 + 2\pi$ and $z(g_1) = z_M + 2\pi\gamma > z_M$, whereas for sufficiently large μ [23], $\mu \geq \tilde{\mu}$, either it is $z(g_1) < z_M$ or even $z(g)$ is not defined in g_1 (as $g_- < g_1$). By continuity in μ and Prop. 2, there exists a unique $\check{\mu}(\gamma, z_M) \in]0, \tilde{\mu}[$ such that the corresponding solution $\check{z}(g) := z(g; g_0, z_M; \check{\mu}, \gamma)$ fulfills the condition (23) [see Fig. 2 right, where also $\check{\mathrm{e}}(g)$ is plotted]. That this holds not only for $k = 0$, but for all k follows from iteration in successive intervals. Actually, by the invariance of (13) under $g \to g + 2\pi$ and the uniqueness of the solution this implies that $\check{z}$ is defined in all $\mathbb{R}$ and periodic:

$$\check{z}(g) = \check{z}(g + 2\pi). \tag{24}$$

By (19) this implies

$$\check{\mu}\, I(\check{z}, g) = 2\pi\gamma; \tag{25}$$

the lhs is independent of g (and can be called simply $\check{I}$). Again this equality amounts to an **energy balance condition**: the energy dissipated by the viscous force equals the potential energy gap after a 2π displacement of the 'particle'. For g fixed $\check{z}$, $\check{I}$ are strictly increasing, continuous functions of z_M by Prop. 1, 2, whereas $\check{\mu}$ and Ξ are strictly decreasing and continuous respectively by (25) and $(8)_2$. All these functions are therefore invertible, and one can adopt any of the four parameters $z_M, \check{I}, \check{\mu}, \Xi$ (in the appropriate range) as the independent one, beside γ. For $|v| < 1$ one can adopt also $|v|$ as the independent parameter, as the function $\mu(|v|)$ defined in $(10)_2$ is strictly monotonic.

Replacing $\check{z}(g)$ in (15) one finds the mentioned $\check{g}(\xi)$, and replacing the latter in (6) one finds one-parameter families of **evenly spaced "arrays of solitons"** and of **evenly spaced "arrays of antisolitons"**, as described in Theorem 1; as a parameter one can choose $z_M, \check{I}, \check{\mu}, \Xi, |v|$. The (anti)soliton solution is recovered as the $z_M \to 0$ limit.

As we are going to see in the next theorem, the $\check{z}(g)$ are asymptotic solutions [23] at

$g\to\infty$, more precisely exponentially attract *all* other solutions $z(g)$ of eq. (13) (with the same value of μ) with domain extending to $g_+=\infty$, in particular, *all* z such that $z>\check{z}$; the latter fact holds also if $\mu=\hat{\mu}$ and we denote by $\check{z}$ the "spurious" periodic solution, i.e. the periodic extension of $\hat{z}$. The $z(g)$ such that $z<\check{z}$ will yield a family of solutions of (1) without unperturbed analog, the "half-array of (anti)solitons".

We have partly proved and are ready to state the following theorem, which includes and completes results partly obtained in Ref. [7].

Theorem 1 *Let*

$$\xi := \frac{x-vt}{\sqrt{1-v^2}}, \qquad\qquad \check{v}(\mu) := \frac{\mu}{\sqrt{\alpha^2+\mu^2}} \leq 1. \tag{26}$$

Mod. 2π, stable[2] travelling-wave solutions of (1) (where $\gamma>0$ and $\alpha\geq 0$) having bounded derivatives at infinity are only of the following types:

1. *The static, uniform solution $\varphi^s(x,t)\equiv -\sin^{-1}\gamma$, if $\gamma<1$.*
2. *The soliton $\hat{\varphi}^+(x,t)=\hat{g}(\xi)-\pi$ and the antisoliton $\hat{\varphi}^-(x,t)=\hat{g}(-\xi)-\pi$,*

   $$\lim_{x\to-\infty}\hat{\varphi}^{\pm}(x,t) = -\sin^{-1}\gamma, \qquad\qquad \lim_{x\to\infty}\hat{\varphi}^{\pm}(x,t) = -\sin^{-1}\gamma\pm 2\pi, \tag{27}$$

   travelling respectively rightwards with velocity $v=\hat{v}:=\check{v}(\hat{\mu}(\gamma))$ and leftwards with velocity $v=-\hat{v}$, only if $\gamma<1$ and $\alpha>0$. The function $\hat{\mu}(\gamma)$ fulfills the bounds (37), is [24] continuous and strictly increasing in $[0,1]$, with $\hat{\mu}(0)=0$; a good numerical determination is in [24], whereby $\mu(1)\approx 1,193$. It can be determined with arbitrary accuracy by the method described in Thm. 2, which gives $\hat{\mu}(\gamma)=\pi\gamma/4+O(\gamma^2)$.
3. *The "array of solitons" $\check{\varphi}^+(x,t;\mu)=\check{g}(\xi;\mu)-\pi$ and the "array of antisolitons" $\check{\varphi}^-(x,t;\mu)=\check{g}(-\xi;\mu)-\pi$, only if $\alpha>0$, and for any $\mu\in]0,\infty]$ if $\gamma>1$, for any $\mu\in]0,\hat{\mu}(\gamma)[$ if $\gamma\leq 1$, where $\check{g}$ fulfills*

   $$\check{g}(\xi+\Xi)=\check{g}(\xi)+2\pi, \qquad\qquad \Xi(\mu,\gamma)=\int\limits_{g}^{g+2\pi}\frac{ds}{\sqrt{2\check{z}(s)}}\in]0,\infty[\tag{28}$$

   ($\check{g}$ is "linear-periodic"), travelling respectively rightwards with velocity $v=\check{v}(\mu)$ and leftwards with velocity $v=-\check{v}(\mu)$. If $\gamma>1$, $\check{\varphi}^{\pm}(x,t;\infty):=g(\pm x-t)-\pi=\lim_{\mu\to\infty}\check{g}\left(\frac{\pm x-|\check{v}|t}{\sqrt{1-\check{v}^2}};\mu\right)-\pi$, with the g found in (8) (these have velocity $v=\pm 1$ respectively).
4. *The "half-array of solitons" $\bar{\varphi}^+(x,t)=\bar{g}(\xi;\mu,\gamma)-\pi$ and the "half-array of anti-solitons" $\bar{\varphi}^-(x,t)=\bar{g}(-\xi;\mu,\gamma)-\pi$ travelling respectively rightwards with velocity $v=\check{v}(\mu)$ and leftwards with velocity $v=-\check{v}(\mu)$, only if $\gamma<1$, $\alpha>0$, and for any $\mu\in]0,\hat{\mu}(\gamma)[$, where $\bar{g}$ fulfills $\bar{g}<\check{g}$ and*

   $$\lim_{x\to\mp\infty}\bar{\varphi}^{\pm}(x,t) = -\sin^{-1}\gamma \qquad\qquad \lim_{g\to\infty}[\bar{z}(g)-\check{z}(g)]=0^- \tag{29}$$

   $$\lim_{\xi\to\infty}[\bar{g}(\xi)-\check{g}(\xi)]=0^+, \qquad\qquad \lim_{\xi\to\infty}[\bar{g}'(\xi)-\check{g}'(\xi)]=0^- \tag{30}$$

[2] The stability has been tested numerically; an analytic study will be done elsewhere. We just note that the key property used in the stability proof of [19], $g'(\xi)>0\ \forall\xi\in\mathbb{R}$, is fulfilled by the families of solutions 2,3,4.

(for an appropriate choice of $\breve{g}$ in the family of $\breve{g}$'s differing only by a ξ-translation). The last three limits are approached exponentially.

All $\hat{g},\breve{g},\bar{g},\bar{g}-\breve{g}$ are strictly increasing. To parameterize the solutions of classes 3,4 one can adopt as an independent variable alternative to μ either $z_M,\check{I},|v|$ or Ξ.

Rest of the proof: In Ref. [7] we showed that solutions g of (9) yielding solutions φ of (1) which have bounded energy density h and are not manifestly unstable *can* only be of the above type. We have just shown that the first three families of solutions *actually exist*, and added some details to their properties. Formula (26) follows from inverting $(10)_2$ in the branch $|v|<1$ (whereas the branch $|v|>1$ yields no relevant solutions). Let $\tilde{g}(\eta):=\breve{g}(\xi)$ with $\eta:=\sqrt{1-v^2}\xi=\mathrm{sign}(v)x-|v|t$. $\tilde{g}_\eta,\tilde{g}_{\eta\eta}$ are periodic. For $\gamma>1$, replacing in (9) and letting $|v|\uparrow 1$ we find that $\tilde{g}$ fufills (7). This proves the limit $\lim_{\mu\to\infty}\breve{g}\left(\frac{\pm x-|\breve{v}|t}{\sqrt{1-\breve{v}^2}};\mu\right)=g(\pm x-t)$, after noting that by $(10)_2$ $\mu\to\infty$ as $|v|\uparrow 1$.

As for the fourth family, consider a $z(g)$ with domain extending to $g_+=\infty$. Since the two diagrams $z(g),\breve{z}(g)$ do not intersect, $w(g):=z(g)-\breve{z}(g)$ is either positive- or negative-definite. By (13) it fulfills

$$w_g=-\mu\left[\sqrt{2(\breve{z}+w)}-\sqrt{2\breve{z}}\right]=-2\mu\frac{w}{\sqrt{2(\breve{z}+w)}+\sqrt{2\breve{z}}},$$

implying

$$\frac{d}{dg}\ln|w|=-\frac{2\mu}{\sqrt{2(\breve{z}+w)}+\sqrt{2\breve{z}}}\leq-\frac{\mu}{\sqrt{2(\breve{z}^M+|w(g_0)|)}}$$

(we have denoted by $\breve{z}^M$ the maximum of $\breve{z}$): $|w(g)|$ is strictly decreasing. By integration we find for $g\geq g_0$

$$|w(g)|\leq|w(g_0)|e^{-C(g-g_0)},\qquad\qquad C:=\frac{\mu}{\sqrt{2(\breve{z}^M+|w(g_0)|)}},\tag{31}$$

namely $|w(g)|\to 0$ exponentially as $g\to\infty$, as claimed.

In the case $\gamma\leq 1$, $\mu<\hat{\mu}(\gamma)$ this applies in particular to the solution $\bar{z}$ fulfilling the initial condition $\bar{z}(g_k^M)=0$ (for some $k\in\mathbb{Z}$). Since $\mu<\hat{\mu}(\gamma)$, by Prop. 2 this is defined and larger than $\hat{z}$, therefore positive in g_{k+1}^M and then by Prop. 3 will be defined in all $G=]g_k^M,\infty[$. Since $A_k=(g_k^M,0)$ is a saddle point, the corresponding path $(\bar{g}(\xi),\bar{u}(\xi))$ is a separatrix and

$$\bar{g}(\xi)\stackrel{\xi\to-\infty}{\longrightarrow}g_k^M,\qquad\qquad\bar{g}(\xi)\stackrel{\xi\to\infty}{\longrightarrow}\infty.\tag{32}$$

As $\breve{z}(g_k^M)>0$, the corresponding $\bar{w}=\bar{z}-\breve{z}$ is negative-definite, and we find in the order

$$\bar{w}:=\bar{z}-\breve{z}\uparrow 0,\qquad\sqrt{2\bar{z}}-\sqrt{2\breve{z}}\uparrow 0,\qquad\frac{1}{\sqrt{2\bar{z}}}-\frac{1}{\sqrt{2\breve{z}}}\downarrow 0,\tag{33}$$

exponentially as $g\to\infty$. The first limit gives $(29)_2$. From (15)

$$\bar{\xi}(g)=\int_{g_0}^{g}\frac{ds}{\sqrt{2\bar{z}(s)}}+\bar{c},\qquad\qquad\breve{\xi}(g)=\int_{g_0}^{g}\frac{ds}{\sqrt{2\breve{z}(s)}}+\breve{c},$$

where $\bar{c}, \check{c}$ are integration constants, whence

$$\bar{\xi}(g) - \check{\xi}(g) = \int_{g_0}^{g} ds \left[\frac{1}{\sqrt{2\bar{z}(s)}} - \frac{1}{\sqrt{2\check{z}(s)}}\right] + (\bar{c} - \check{c}).$$

The integrand is positive and goes exponentially to zero as $g \to \infty$, therefore the integral converges. Choosing the constants so that $\int_{g_0}^{\infty} ds[] = (\check{c} - \bar{c})$ we find

$$\bar{\xi}(g) = \check{\xi}(g) - \rho(g), \qquad \rho(g) := \int_{g}^{\infty} ds \left[\frac{1}{\sqrt{2\bar{z}(s)}} - \frac{1}{\sqrt{2\check{z}(s)}}\right]$$

with $\rho(g)$ positive and exponentially vanishing. Applying the inverse $\bar{g}(\xi)$ of $\bar{\xi}(g)$ to both sides we find

$$g = \bar{g}\left(\check{\xi}(g) - \rho(g)\right) = \bar{g}\left(\check{\xi}(g)\right) - \bar{g}'(\tilde{\xi})\rho(g).$$

The second equality is based on Lagrange theorem, where $\tilde{\xi}$ is a suitable point in $]\check{\xi}(g) - \rho(g), \check{\xi}(g)[$. Finally, setting $g = \check{g}(\xi)$ we find

$$\check{g}(\xi) = \bar{g}(\xi) - \bar{g}'(\tilde{\xi})\rho(\check{g}(\xi)),$$

where $\tilde{\xi} \in]\xi - \rho(\check{g}(\xi)), \xi[$. The second term at the rhs exponentially vanishes as $\xi \to \infty$ [since $\rho(\check{g}(\xi))$ does and $\bar{g}'$ is bounded], proving $(30)_1$. By $(30)_1$ the second limit in (33) now implies $(30)_2$.

$\square$

Remark 3.1 In Fig. 3 we have plotted a soliton, array of solitons and "half-array of (anti)solitons" solutions, determined numerically. The latter has no unperturbed analog. It interpolates between the (anti)soliton solution at one extreme and the array of (anti)solitons solution at the other. Therefore it cannot be approximated, nor can it even be figured out, by the modulation Ansatz (2).

Remark 3.2 Setting $X := \Xi\sqrt{1 - v^2}$, eq. (28) and (6) imply

$$\check{\varphi}^{\pm}(x + X, t) = \check{\varphi}^{\pm}(x, t) \pm 2\pi; \tag{34}$$

so $\check{\varphi}^{\pm}$ makes sense also as a solution of (1) **on a circle of length** $L = mX$, for any $m \in \mathbb{N}$. The integer m parameterizes different topological sectors: in the m-th the pendula chain twists around the circle m times.

Remark 3.3 We emphasize that, in contrast with the unperturbed soliton (and array of solitons) solutions, where v was a free parameter of modulus less than 1, v is predicted as a function of γ, α for the perturbed soliton, as a function of γ, α and one of the parameters $z_M, \check{I}, \Xi$ for the perturbed (half-)array of solitons.

We determine the ranges of the various parameters. Clearly, as $z_M \to \infty$ $\check{z}$ and $\check{I}$ diverge, whereas $\check{\mu}, \Xi, \check{v}$ go to zero.
We now consider the limit $z_M \to 0$. If $\gamma > 1$, as $z_M \to 0$ one finds the following leading parts and limits

$$\begin{aligned} &\check{\mu} \approx \tfrac{\gamma-1}{\sqrt{2z_M}} \to \infty, && \check{I} \approx \tfrac{\sqrt{2z_M}2\pi\gamma}{\gamma-1} \to 0, \\ &\Xi \sim \tfrac{1}{\sqrt{z_M}} \to \infty, && \check{v} \approx \tfrac{\gamma-1}{\sqrt{2z_M\alpha^2+(\gamma-1)^2}} \to 0; \end{aligned} \tag{35}$$

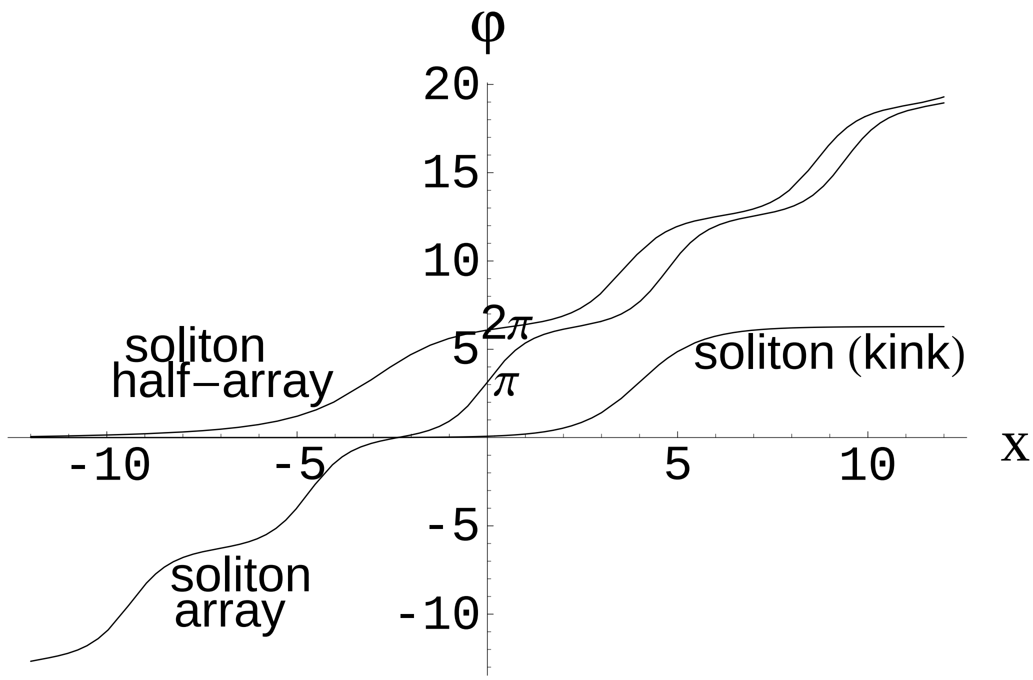

Figure 3: The soliton, the array of solitons and the half-array of solitons solutions

as z_M spans $]0,\infty[$, the range of any of $\check{I},\check{\mu},\Xi$ is $]0,\infty[$ and that of $\check{v}$ is $]0,1[$. In fact, the Taylor formula of second order for $\check{z}(g)$ around g_k can be written without loss of generality in the form

$$\check{z}(g;z_M;\gamma)=z_M+z_M\zeta_1(z_M;\gamma)(g-g_k)+(g-g_k)^2\rho(g) \tag{36}$$

with $\rho(g)$ bounded; in order that, as $z_M\to 0$, $\check{z}$ keeps nonnegative both in a left and a right neighbourhood of g_k, $\zeta_1(z_M;\gamma)$ has to approach a *finite* limit. Replacing this Ansatz in (13) we find at lowest order in $(g-g_k)$

$$z_M\zeta_1+\check{\mu}\sqrt{2z_M}+1-\gamma=0.$$

As $z_M\to 0$ this implies (35) [by (25), $(28)_2$ and $(26)_2$] . Summarizing, as z_M spans $]0,\infty[$ the range of any of $\check{I},\check{\mu},\Xi$ is $]0,\infty[$ and that of $\check{v}$ is $]0,1[$.
If $\gamma\leq 1$, by the monotonicity property $\check{\mu}(\gamma,z_M)\leq\hat{\mu}(\gamma)$, and by the continuity we find [23]

$$\lim_{z_M\to 0}\check{\mu}(\gamma,z_M)=\hat{\mu}(\gamma)<\infty.$$

Hence if $\gamma\leq 1$ the range of $\check{\mu}$ as z_M spans $[0,\infty[$ is $]0,\hat{\mu}]$, the range of $\check{I}$ is $]2\pi\gamma/\mu,\infty[$ the range of v is $[0,\hat{v}[$. The following bounds for $\hat{\mu}(\gamma)$ have been proved ([23, 9], see [17] for a summary)

$$\sqrt{\sqrt{3(1-\gamma^2)+1}-2\sqrt{1-\gamma^2}}\leq\hat{\mu}(\gamma)\leq\sqrt{2\left(1-\sqrt{1-\gamma^2}\right)}.,\tag{37}$$

4 Method of successive approximations

Eq. (17) can be reformulated as the fixed point equation

$$Az = z \tag{38}$$

for $z(g)$, where for $\epsilon > 0$ the operator $A = A(g_0, z_0; \mu, \gamma)$ is defined by

$$\begin{aligned} Aw(g) &:= \omega(g; g_0, z_0; \gamma) - \int_{g_0}^{g} ds\, \phi(g, s, w(s)) \\ \omega(g; g_0, z_0; \gamma) &:= z_0 + U(g_0) - U(g) \qquad \phi(g, s, \zeta) := \sqrt{2\zeta}\mu \end{aligned} \tag{39}$$

on the space of nonnegative smooth functions w on $\mathbb{R}$ (the domain of w can be always trivially extended to $\mathbb{R}$). According to the method of successive approximations, after a reasonable choice of a function $z_{(0)}(g)$ as an initial approximation for $z(g)$, better and better approximations should be provided by $z_{(n)} := A^n z_{(0)}(g)$ as $n \to \infty$. For this to make sense, at each step it is necessary that $z_{(n)}$ belongs to the domain of A (in the present case, it must be nonnegative, otherwise the integrand function is ill-defined) and that the sequence converges. *With the known standard theorems*, this can be guaranteed a priori not in the whole domain G of the unknown z, but only in some smaller interval J containing g_0. In general only the iterated application in infinitely many adjacent intervals allows to extend a local solution to a global one, what makes the procedure of little use for its concrete determination.

Estimating the length of such a J one finds that it is not less than 2π only for sufficiently large z_0. Actually, the determination of the solution in an interval of length 2π would be enough for the complete determination both in the case of a periodic solution $\check{z}$ (which is then extended periodically) and of a separatrix $\hat{z}$ (in that case $G =]g^M_{k-1}, g^M_k[$, which has exactly length 2π). The periodicity condition (23) is automatically fulfilled by each $z_{(n)}$ if we modify the definition of A adjusting the coefficient μ to w as follows:

$$\tilde{A}\, w := A(g_0, z_0; \tilde{\mu}(w), \gamma)\, w \qquad\qquad \tilde{\mu}(w) := 2\pi\gamma \left[\int_{g_0}^{g_0+2\pi} ds \sqrt{2w(s)} \right]^{-1} \tag{40}$$

Choosing $g_0 = g^M_{k-1}$ for simplicity, then $\tilde{\mu}(z_{(n)})$ will converge to $\check{\mu}(\gamma, z_0)$. If instead we fix μ as an independent parameter, one will obtain z_0 as $\lim_n z_{(n)}(g_0)$ [23]. For the periodic solution a sufficiently large z_0 amounts to a sufficiently small μ; in [23] the following quantitative condition was found:

$$\eta_1 > \epsilon_1, \qquad \mu < \frac{(\sqrt{\eta_1} - \sqrt{\epsilon_1})^2}{2\pi\sqrt{2}} \tag{41}$$

where

$$\epsilon_1 := \max |z_{(1)} - z_{(0)}| \equiv \|z_{(1)} - z_{(0)}\|_\infty, \qquad \eta_1 := \min |z_{(1)}|.$$

So η_1 cannot be too small, in particular cannot vanish, what excludes the cases of the periodic solutions $\check{z}$ having low energy and of the heteroclinic orbit $\hat{z}$.

4.1 The soliton solution by the method of successive approximations

The standard theorems fail for $\hat{z}$ because the sup norm has not enough control to guarantee non-negativity of the approximations $z_{(n)}$ everywhere in G, as well as the fulfillment of a Lipschitz condition by the integrand ϕ and the behaviour (16) near the extremes of G. In this section *we adopt a tricky, nonstandard choice of the norm and show (Theorem 2) that a single application of the method of successive approximations gives the soliton solution* $(\hat{\mu}, \hat{z}(g))$ *in its whole domain* $G =]g_{k-1}^M, g_k^M[$.

Assume $\gamma < 1$. Choose $g_0 = g_{k-1}^M$, $z_0 = 0$ and let $y := g - g_0$. Then

$$\omega(y) = \sqrt{1-\gamma^2} 2 \sin^2 \frac{y}{2} + \gamma(y - \sin y) = \frac{1}{2}\sqrt{1-\gamma^2} y^2 + O(y^3) \tag{42}$$

and $\hat{z}$ fulfills (38), where the operator $\tilde{A}$ has taken the form

$$\tilde{A}z(y) \equiv \tilde{z}(y) := \sqrt{1-\gamma^2} 2 \sin^2 \frac{y}{2} + \gamma(y - \sin y) - \tilde{\mu}(z) \int_0^y dy' \sqrt{2z(y')},$$
$$\text{where } \tilde{\mu}(z) := \frac{2\pi\gamma}{\int_0^{2\pi} dy' \sqrt{2z(y')}} \tag{43}$$

By (16) $\hat{z}(y) = O(y^2)$, $\hat{z}(2\pi - y) = O\Big((2\pi - y)^2\Big)$. One easily checks that, more generally, if z has such a behaviour near $0, 2\pi$, so has $\tilde{A}z$. So it would be more natural to look for the solution from the very beginning in a functional space whose elements have such a behaviour. In $C^1([0, 2\pi])$ introduce the norm

$$\|z\| = \sup_{y \in [0,2\pi]} \left| \frac{2z(y)}{p^2(y)} \right| \tag{44}$$

where the 'weight' p should vanish as y and $2\pi - y$ at $0, 2\pi$ and will be specified later. Clearly

$$\|z\| \geq C \|z\|_\infty \equiv C \sup_{y \in]0,2\pi[} |z(y)| \qquad C^{-1} := \sup_{y \in]0,2\pi[} \frac{p^2(y)}{2}. \tag{45}$$

The subspace

$$V := \{z(y) \in C^\infty([0, 2\pi]) \mid \|z\| < \infty\} \tag{46}$$

is a complete metric space w.r.t. the metric induced by the above norm. In fact, consider a Cauchy sequence $\{z_n\} \subset V$ in the norm $\|\cdot\|$: by (45) it is Cauchy and therefore converges to a (uniformly continuous) function $z(y)$ also in the norm $\|\cdot\|_\infty$; moreover for any $\varepsilon > 0$ there exists $\bar{r} \in \mathbb{N}$ such that $\forall r \geq \bar{r}$, $\forall m \in \mathbb{N}$

$$\sup_{y \in [0,2\pi]} \left| \frac{z_r(y) - z_{r+m}(y)}{p^2(y)} \right| < \frac{\varepsilon}{2};$$

Letting $m \to \infty$ we find

$$\sup_{y \in [0,2\pi]} \left| \frac{z_r(y) - z(y)}{p^2(y)} \right| < \varepsilon,$$

showing that $z \in V^3$ and that $\{z_n\} \to z$ also w.r.t. the topology induced by the above norm.

Let $a, b \in \mathbb{R}$ with $b > a > 0$. The subset

$$Z_{a,b,p} := \left\{ z(y) \in V \;|\; a^2 \le \frac{2z(y)}{p^2(y)} \le b^2 \right\} \tag{47}$$

is clearly closed w.r.t. the metric induced by the above norm. We shall look for $(\hat{z}, \hat{\mu})$ within a suitable $Z_{a,b,p}$. First we look for a, b such that (43) defines an operator $\tilde{A} : Z_{a,b,p} \to Z_{a,b,p}$. Up to a factor, we choose $p^2(y)$ as the $\gamma = 0$ (i.e. unperturbed) soliton solution $\hat{z}_0(y)$, more precisely $p(y) := \sin \frac{y}{2}$. Then

$$P(y) := \int\limits_0^y dy' p(y') = 2(1 - \cos \frac{y}{2}) = \int\limits_y^{2\pi} dy' p(y'),$$

and, since $1 - \sqrt{1 - w} \ge w/2$ we find (setting $w = \sin^2 \frac{y}{2}$)

$$p^2(y) \le P(y) \le 2 \left(1 - \cos \frac{y}{2}\right) \left(1 + \cos \frac{y}{2}\right) = 2p^2(y).$$

Thus for any $z \in Z_{a,b,p}$ we find

$$aP(y) \le \int\limits_0^y dy' \sqrt{2z(y')} = \int\limits_0^y dy' \frac{\sqrt{2z(y')}}{p(y')} p(y') \le bP(y)$$

$$4a = aP(2\pi) \le \frac{2\pi\gamma}{\tilde{\mu}} = \int\limits_0^{2\pi} dy' \sqrt{2z(y')} = \le bP(2\pi) = 4b$$

implying the inequalities $\gamma\pi/2b \le \tilde{\mu} \le \gamma\pi/2a$ and

$$\gamma \frac{\pi a}{2b} p^2(y) \le \tilde{\mu} \int\limits_0^y dy' \sqrt{2z(y')} \le \gamma \frac{\pi b}{a} p^2(y). \tag{48}$$

Similarly,

$$\gamma \frac{\pi a}{2b} p^2(y) \le \tilde{\mu} \int\limits_y^{2\pi} dy' \sqrt{2z(y')} \le \gamma \frac{\pi b}{a} p^2(y). \tag{49}$$

Lemma 1 *For all $y \ge 0$*

$$1 - \cos y \ge 0, \qquad y - \sin y \ge 0, \qquad \frac{y^2}{2} - 1 + \cos y \ge 0, \qquad \frac{y^3}{6} - y + \sin y \ge 0.$$

[3]If *per absurdum* $\sup |z/p^2| = \infty$ then the lhs would certainly exceed ε.

Proof: The first equality is obvious; the others are obtained by iterated integration over $[0,y]$. Q.E.D.

As a consequence, for $y \in [0,\pi]$

$$0 \leq y - \sin y \leq \frac{y^3}{6} = \frac{1}{6} p^2(y) \left[\frac{y}{\sin \frac{y}{2}}\right]^2 y \leq p^2(y) \frac{\pi^3}{6}. \tag{50}$$

Collecting the results, on one hand assuming $1 \geq a/b \geq 1/2$ we find

$$\tilde{z}(y) \geq p^2(y) \left[2\sqrt{1-\gamma^2} - \gamma\pi \frac{b}{a}\right] \geq p^2(y) 2 \left[\sqrt{1-\gamma^2} - \gamma\pi\right] \tag{51}$$

for all $y \in [0, 2\pi]$; on the other hand, for $y \in [0,\pi]$ we find

$$\tilde{z}(y) \leq p^2(y) 2 \left[\sqrt{1-\gamma^2} + \gamma \frac{\pi^3}{12}\right]. \tag{52}$$

This provides bounds for $y \in [0,\pi]$. To find bounds for $y \in [\pi, 2\pi]$ set $v = (2\pi - y)$ and note that from (43) it follows

$$\begin{aligned}
\tilde{z}(y) &= \sqrt{1-\gamma^2}\, 2\sin^2 \frac{y}{2} - \gamma(v - \sin v) + 2\pi\gamma - \tilde{\mu} \left[\int_0^{2\pi} dy \sqrt{2z} - \int_y^{2\pi} dy' \sqrt{2z(y')}\right] \\
&= \sqrt{1-\gamma^2}\, 2\sin^2 \frac{y}{2} - \gamma(v - \sin v) + \tilde{\mu} \int_y^{2\pi} dy' \sqrt{2z(y')},
\end{aligned}$$

We use (49) to bound the third term at the rhs; as $v \in [0,\pi]$, to bound the second term we can use (50) with y replaced by v, but keeping $p^2(y) = p^2(v)$ at the rhs of the latter. Collecting the results we thus find for $y \in [\pi, 2\pi]$

$$p^2(y) 2 \left[\sqrt{1-\gamma^2} - \gamma \frac{\pi^3}{12}\right] \leq \tilde{z}(y) \leq p^2(y) 2 \left[\sqrt{1-\gamma^2} + \gamma\pi\right]. \tag{53}$$

Hence $a^2 p^2 \leq 2\tilde{z} \leq b^2 p^2$, so that $\tilde{z} \in Z_{a,b,p}$, if we define

$$a^2 := 4\left[\sqrt{1-\gamma^2} - \gamma\pi\right] \qquad b^2 := 4\left[\sqrt{1-\gamma^2} + \gamma\pi\right]. \tag{54}$$

In order that $1/2 \leq a/b$ it must be

$$\frac{1}{4} \leq \frac{a^2}{b^2} = \frac{\sqrt{1-\gamma^2} - \gamma\pi}{\sqrt{1-\gamma^2} + \gamma\pi}$$

which gives, after some computation,

$$\gamma \leq \left[1 + \frac{25\pi^2}{9}\right]^{-\frac{1}{2}} \approx .187 \tag{55}$$

We conclude that in this γ-range with the above choice of a, b $\tilde{A} Z_{a,b,p} \subset Z_{a,b,p}$, as required.

Let us determine the constraints on a, b following from the condition that $\tilde{A}$ be a contraction. First, we immediately find

$$2|z_1(y) - z_2(y)| = p^2(y)\frac{2|z_1(y) - z_2(y)|}{p^2(y)} \leq p^2(y)\|z_1 - z_2\|$$

Note that for any $\alpha > 0$, $|\sqrt{u_1} - \sqrt{u_2}| \leq |u_1 - u_2|/(2\alpha)$ if $u_1, u_2 \in [\alpha^2, \infty[$. Hence

$$\begin{aligned} |\sqrt{2z_1(y)} - \sqrt{2z_2(y)}| &= p(y)\left|\sqrt{\frac{2z_1(y)}{p^2(y)}} - \sqrt{\frac{2z_2(y)}{p^2(y)}}\right| \leq \frac{p(y)}{2a}\frac{2|z_1(y) - z_2(y)|}{p^2(y)} \\ &\leq \frac{p(y)}{2a}\|z_1 - z_2\| \end{aligned} \tag{56}$$

$$\begin{aligned} |\tilde{\mu}_1 - \tilde{\mu}_2| &= \tilde{\mu}_1\tilde{\mu}_2|\tilde{\mu}_1^{-1} - \tilde{\mu}_2^{-1}| \leq \frac{\pi\gamma}{8a^2}\left|\int_0^{2\pi} dy(\sqrt{2z_1(y)} - \sqrt{2z_2(y)}\right| \\ &\leq \frac{\pi\gamma}{8a^2}\int_0^{2\pi} dy\left|\sqrt{2z_1(y)} - \sqrt{2z_2(y)}\right| \leq \frac{\pi\gamma}{16a^3}\|z_1 - z_2\|\int_0^{2\pi} dy p(y) \\ &= \frac{\pi\gamma}{4a^3}\|z_1 - z_2\| \end{aligned} \tag{57}$$

$$\tilde{z}_2 - \tilde{z}_1 = \int_0^y dy'\left[(\tilde{\mu}_1 - \tilde{\mu}_2)\sqrt{2z_1(y')} + \tilde{\mu}_2(\sqrt{2z_1(y')} - \sqrt{2z_2(y')})\right]$$

whence

$$\begin{aligned} |\tilde{z}_1(y) - \tilde{z}_2(y)| &\leq |\tilde{\mu}_1 - \tilde{\mu}_2|\int_0^y dy' p(y')\sqrt{\frac{2z_1(y')}{p^2(y')}} + \tilde{\mu}_2\int_0^y dy'\left|\sqrt{2z_1(y')} - \sqrt{2z_2(y')}\right| \\ &\leq \frac{\pi b\gamma}{4a^3}\|z_1 - z_2\|P(y) + \frac{\pi\gamma}{4a^2}\|z_1 - z_2\|P(y) \\ &\leq \left(1 + \frac{b}{a}\right)\frac{\pi\gamma}{4a^2}\|z_1 - z_2\|P(y) \leq \left(1 + \frac{b}{a}\right)\frac{\pi\gamma}{2a^2}\|z_1 - z_2\|p^2(y), \end{aligned}$$

implying

$$\|\tilde{z}_1(y) - \tilde{z}_2(y)\| \leq \left(1 + \frac{b}{a}\right)\frac{\pi\gamma}{a^2}\|z_1 - z_2\|. \tag{58}$$

Thus, $\tilde{A}$ is a contraction if

$$\lambda := (1 + b/a)\pi\gamma/a^2 < 1, \tag{59}$$

that is,

$$\gamma < \frac{a^2}{\pi\left(1 + \frac{b}{a}\right)} \leq \frac{a^2}{3\pi} = \frac{4}{3\pi}[\sqrt{1-\gamma^2} - \gamma\pi],$$

namely if

$$\gamma < \left[1+\left(\frac{7\pi}{4}\right)^2\right]^{-\frac{1}{2}} \approx .179 \tag{60}$$

Summing up, under this condition $\tilde{A}$ is a contraction of $Z_{a,b,p}$ into itself. Since $z_{(0)}(y) := 2p^2(y) = 2\sin^2\frac{y}{2}$ belongs to $Z_{a,b,p}$, applying the Banach fixed point theorem we find

Theorem 2 *Let $z_{(0)}(y) := 2\sin^2\frac{y}{2}$, $z_{(n)} := \tilde{A}^n z_{(0)}$, $\mu_n := \tilde{\mu}(z_{(n-1)})$, with $\tilde{A},\tilde{\mu}$ defined as in (43). The sequences $\{z_{(n)}\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ converge respectively to the soliton solution $\hat{z}$ [in the norm (44)] and to the corresponding $\hat{\mu}(\gamma)$, for γ at least in the range (60). With λ defined as in (59), the errors of the n-th approximation are bound by*

$$\left\|z_{(n)} - \hat{z}\right\| \leq \frac{\lambda^n}{1-\lambda}\left\|z_{(1)}-z_{(0)}\right\|, \qquad |\mu_n-\hat{\mu}| \leq \frac{\pi\gamma}{32}\left[\sqrt{1-\gamma^2}-\gamma\pi\right]^{-\frac{3}{2}}\frac{\lambda^n}{1-\lambda}\left\|z_{(1)}-z_{(0)}\right\| \tag{61}$$

[To complete the proof we need just to note that, by (57), the convergence of $z_{(n)}$ implies the convergence of μ_n and estimate the second error through standard arguments].

Remark 4.1 More refined computations of upper and lower bounds, with the present γ-independent weight $p^2(y) = \sin^2\frac{y}{2}$, would show a γ-range of convergence of the above sequences slightly larger than (60). By choosing a suitable γ-dependent weight $p^2(y)$, e.g. $p^2(y) = z_{(1)}(y)/2$, one could show that this range is actually significantly larger. This will be elaborated elsewhere.

We explicitly work out the first approximation. We find:

$$z_{(1)}(y) = \sqrt{1-\gamma^2}2\sin^2\frac{y}{2}+\gamma\left[\pi\left(\cos\frac{y}{2}-1\right)+y-\sin y\right] \tag{62}$$

$$\mu_1 = \frac{1}{4}\pi\gamma \tag{63}$$

$$\mathrm{e}_{(1)}(y) = \gamma\pi\left(\cos\frac{y}{2}-1\right)+\mathrm{const} \tag{64}$$

$$v_{(1)}(\gamma,\alpha) = \frac{1}{\sqrt{1+(4\alpha/\pi\gamma)^2}} = \frac{\pi\gamma}{4\alpha}+O(\gamma^2). \tag{65}$$

The results are in good agreement with the numerical simulation plot in Fig. 2 (right). Note that the result (65) coincides with (3), as announced. In a similar way one can determine iteratively solutions of type 3 $(\mu,\check{z})$ even with low z_M [i.e. not fulfilling the bound (41].

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Appendix

Classification of the singular points of (11). It is easy to check that the characteristic equations of (11) are

$$\lambda^2 + \mu\lambda \mp \sqrt{1-\gamma^2} = 0; \tag{66}$$

the upper, lower sign refer to any A_k, B_k respectively, $\gamma = 1$ to C_k.

- The solutions λ_1, λ_2 for A_k are both real, and distinct. A_k is a saddle point and there are exactly four half-paths (called separatrices) with an endpoint on A_k: the two ingoing represent motions approaching A_k as $\xi \to \infty$ from the left or from the right, the two outgoing represent motions leaving from A_k as $\xi \to -\infty$ towards the left or towards the right.
- The solutions λ_1, λ_2 for B_k are:
  - Both real if $\mu \geq 2(1-\gamma^2)^{1/4}$. B_k is a node, and there are an infinite number of half-paths ingoing to B_k with the same tangent. These represent overdamped motions ending in B_k as $\xi \to \infty$.
  - Complex conjugates (but not purely imaginary) if $0 < \mu < 2(1-\gamma^2)^{1/4}$. B_k is a focus, and there are an infinite number of half-paths ingoing to B_k along a spiral. These represent damped oscillatory motions ending in B_k as $\xi \to \infty$.
  - Opposite imaginary if $\mu = 0$. Any B_k is a center, and there exist closed paths (cycles) encircling it. These represent periodic motions around B_k, i.e. periodic oscillatory motion of the particle around g_k^m.
- If $\gamma = 1$ $\lambda_1 = 0$, $\lambda_2 = -\mu$ and if $\mu > 0$ C_k is a saddle-node: there only two half-paths (separatrices) in the half-plane $g > (2k + 1/2)\pi$ (the ingoing represents a motion approaching A_k as $\xi \to \infty$ from the right, the outgoing a motion leaving from A_k towards the right as $\xi \to -\infty$) and infinitely many in the half-plane $g < (2k + 1/2)\pi$ (overdamped motions coming from the left and ending in C_k).

Proof of Prop. 1. Let $0 \leq z_{0,2} < z_{0,1}$, $z_j(g) := z(g; g_0, z_{0,j}; \mu, \gamma)$ $(j = 1, 2)$ be the corresponding solutions of (13) and G_j the corresponding intervals giving their (maximal) domains. By continuity the inequality

$$z_1 - z_2 > 0 \tag{67}$$

will hold in a neighbourhood of g_0 within $G_1 \cap G_2$. In fact, it will hold for all $g \in G_1 \cap G_2$. If *per absurdum* this were not the case, denote by $\bar{g} \in G_1 \cap G_2$ the least $g > g_0$ (resp. largest $g < g_0$) where $z_1 - z_2$ vanishes: $z_1(\bar{g}) - z_2(\bar{g}) = 0$; then the problem (13) with initial (resp. final) condition $z(\bar{g}) = z_1(\bar{g}) \equiv z_2(\bar{g})$ would admit the two different solutions z_1, z_2, against the existence and uniqueness theorem. As for the monotonicity of $g_\pm$, by the same theorem $z_1(g_{2\pm}) > z_2(g_{2\pm}) = 0$ implies $g_{1+} > g_{2+}$ if $g_{2+} < \infty$, otherwise $g_{1+} = g_{2+} = \infty$, and $g_{1-} < g_{2-}$ if $g_{2-} > -\infty$, otherwise $g_{1-} = g_{2-} = -\infty$.

Proof of Prop. 2. Let $\mu_1 \leq \mu_2$, $\gamma_1\epsilon \geq \gamma_2\epsilon$, with one of the two inequalities being strict; for $j = 1, 2$ let $u_j(g) := u(g; g_0, u_0; \mu_j, \gamma)$ be the corresponding solutions of (13) with the

same condition $u_j(g_0) = u_0$, and G_j the intervals giving their (maximal) domains. We find

$$u_{2g} = -\mu_2 + \frac{\gamma_2 - \sin g}{u_2} < -\mu_1 + \frac{\gamma_1 - \sin g}{u_2}.$$

By the comparison principle[4] (see e.g. [25]) it follows, as claimed,

$$u_1(g) > u_2(g) \quad g \in]g_0, g_+[, \qquad\qquad u_1(g) < u_2(g) \quad g \in]g_-, g_0[. \tag{68}$$

If $\epsilon > 0$, this implies: $\lim_{g\downarrow g_{2+}} u_1(g) \geq \lim_{g\downarrow g_{2+}} u_2(g) = 0$ and therefore $g_{1+} \geq g_{2+}$ (the inequalities being strict as long as $g_{2+} < \infty$); $\lim_{g\uparrow g_{1-}} u_2 \geq 0$ and therefore $g_{1,-} \geq g_{2-}$ (the inequalities being strict as long as $g_{1-} > -\infty$). Moreover, let $g_j(\xi) = g(\xi; g_0, u_0; \mu_j, \gamma_j)$ be the corresponding two solutions of (14), i.e. the solutions of (9). We find

$$g_2'(\xi) = u_2(g_2(\xi)) \begin{cases} < u_1(g_2(\xi)), & \forall \xi > \xi_0, \\ > u_1(g_2(\xi)), & \forall \xi < \xi_0, \end{cases}$$

while $g_2(\xi_0) = g_0 = g_1(\xi_0)$. By the comparison principle this implies as claimed $g_2(\xi) < g_1(\xi)$ for all $\xi \in X_1 \cap X_2$. Similarly one argues if $\epsilon < 0$.

Proof of Prop. 3. Consider the Cauchy problem (13) in subsequent intervals $]g_k, g_{k+1}[\subset G$. Since the equation is invariant under $g \to g+2\pi$, by Prop. 1 if $z(g_1)$ is respectively larger, equal, smaller than $z(g_0)$ then so are $z(g_{k+1}), I_{k+1}$ in comparison with $z(g_k), I_k$ respectively, for all $k \in K$; in other words, the sequences $\{z(g_k)\}$, $\{I_k\}$ are either constant, or strictly monotonic. Eq. (19) follows from (17) applied in $]g_k, g_{k+1}[$.

If $\epsilon = -$, then rhs(19)$> 2\pi\gamma > 0$ for any k, so that the sequences are strictly increasing and diverging as $k \to \infty$, whereas K must have a lower bound, otherwise $z(g_k)$ would become negative for sufficiently low k.

If $\epsilon = +$, then the two terms at the rhs(19) have opposite sign and can compensate each other. If the sequences are strictly increasing, the sides of (19) are positive for all k and $I_k < 2\pi\gamma/\mu$. Applying (17) to the interval $[g_k, g_k+\Delta g]$ for any $\Delta g \leq 2\pi$ we find

$$z(g_k+\Delta g) - z(g_k) = U(g_k) - U(g_k+\Delta g) - \mu \int_{g_k}^{g_k+\Delta g} ds\sqrt{2z(s)}.$$

But $|U(g_k) - U(g_k+\Delta g)|$ is upper bounded, e.g. by $2+2\pi\gamma$, whence

$$|z(g_k+\Delta g) - z(g_k)| \leq 2+2\pi\gamma + \mu I_k < 2+4\pi\gamma. \tag{69}$$

If *per absurdum* $z(g_k)$ diverged as $k \to \infty$, then also $z(g_k+\Delta g)$ and in turn I_k [by (18)] would diverge, in contrast with $I_k < 2\pi\gamma/\mu$; so it must converge. Moreover, as before, K must have a lower bound. On the other hand, rewriting (19) in the form $z(g_{k-1}) - z(g_k) = \mu I_{k-1} - 2\pi\gamma$, we see that if the sequences $\{z(g_k)\}, \{I_k\}$ are strictly decreasing, the sides are positive for all k and larger than $\mu I_0 - 2\pi\gamma > 0$ for all negative k; this implies that they diverge as $k \to -\infty$, and again by (69) so do $z(g), I(z,g)$. Whereas they must either converge as $k \to \infty$, or K must have an upper bound.

[4]Here we recall the latter in the restricted version: if f fulfills conditions ensuring that the differential problem $\tilde{u}' = f(x, \tilde{u})$, $\tilde{u}(x_0) = u(x_0)$, has a unique solution $\tilde{u}$, and $u' < f(x, u)$ for all x, then it is $u(x) < \tilde{u}(x)$ for all $x > x_0$ and $u(x) > \tilde{u}(x)$ for all $x < x_0$.

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